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Experimental Statistics

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Reprint of the Experimental Statistics Portion of the AMC Handbook

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National Bureau of Standards Handbook 91

Issued August 1, 1963

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C.
Price $4.25

Procedure

Problem: If we are to make a simple series of measurements, how many measurements are required to estimate the standard deviation within P percent of its true value, with prescribed confidence?

(1) Specify P, the allowable percentage deviation of the estimated standard deviation from its true value.

(2) Choose γ, the confidence coefficient.

(3) In Figure 2-2, find P on the horizontal scale, and use the curve for the appropriate γ. Read on the vertical scale the required degrees of freedom.

(4) For a simple series of measurements, the required number is equal to one plus the degrees of freedom.

Example

Problem: How large a sample would be required to estimate the standard deviation within 20% of its true value, with confidence coefficient equal to 0.95?

(1) Let $P = 20\%$

(2) Let $\gamma = .95$

(3) For $\gamma = .95$, $P = 20\%$, the required degrees of freedom equals 46.

(4) $n = 46 + 1$
$= 47$

2-5 STATISTICAL TOLERANCE LIMITS

2-5.1 GENERAL

Sometimes we are more interested in the approximate *range* of values in a lot or population than we are in its average value. Statistical tolerance limits furnish limits between, above, or below which we confidently expect to find a prescribed proportion of individual items of the population. Thus, we might like to be able to give two values A and B between which we can be fairly certain that at least a proportion P of the population will lie, (two-sided limits), or a value A above which at least a proportion P will lie, (one-sided limit).

Thus for the data on thickness of mica washers (Data Sample 2-1), we could give two thickness values, stating with chosen confidence that a proportion P (at least) of the washers in the lot have thicknesses between these two limits. We call the confidence coefficient γ, and it refers to the proportion of the time that our method will result in correct statements. If a normal distribution can be assumed, use the procedures of Paragraphs 2-5.2 and 2-5.3; otherwise use the procedures of Paragraph 2-5.4.

2-5.2 TWO-SIDED TOLERANCE LIMITS FOR A NORMAL DISTRIBUTION

When the mean m and standard deviation σ of a normally distributed quantity are known, symmetrical limits that include a prescribed proportion P of the distribution are readily obtained by adding and subtracting $z_\alpha \sigma$ from the known mean m, where z_α is read from Table A-2 with $\alpha = \frac{1}{2}(P+1)$. When m and σ are not known, we can use an interval of the form $\bar{X} \pm Ks$. Since both $\bar{X}$ and s will vary from sample to sample it is impossible to determine K so that the limits $\bar{X} \pm Ks$ will always include a specified proportion P of the underlying normal distribution. It is, however, possible to determine K so that in a long series of samples from the same or different normal distributions a definite proportion γ of the intervals $\bar{X} \pm Ks$ will include P or more of the underlying distribution(s).

Procedure

Problem: We would like to state two limits between which we are 100 γ percent confident that 100 P percent of the values lie.

(1) Choose P, the proportion, and γ, the confidence coefficient.

(2) Compute from the sample:
$\bar{X}$
s

(3) Look up K for chosen P and γ in Table A-6.

(4) Compute:
$X_U = \bar{X} + Ks$

$X_L = \bar{X} - Ks$

Conclude: With 100 γ % confidence we may predict that a proportion P of the individuals of the population have values between X_L and X_U.

Example

Problem: We would like to state thickness limits between which we are 95% confident that 90% of the values lie (Data Sample 2-1).

(1) Let $P = .90$
$\gamma = .95$

(2)
$\bar{X} = .1260$ inch
$s = 0.00359$ inch

(3) $K = 2.839$

(4)
$X_U = .1260 + 2.839\ (.00359)$
$= 0.136$ inch
$X_L = .1260 - 2.839\ (.00359)$
$= 0.116$ inch

Conclude: With 95% confidence, we may say that 90% of the washers have thicknesses between 0.116 and 0.136 inch.

2-5.3 ONE-SIDED TOLERANCE LIMITS FOR A NORMAL DISTRIBUTION

Sometimes we are interested only in estimating a value above which, or below which, a proportion P (at least) will lie. In this case the one-sided upper tolerance limit will be $X_U = \bar{X} + Ks$; and $X_L = \bar{X} - Ks$ will be the one-sided lower limit. The appropriate values for K are given in Table A-7 and are not the same as those of Paragraph 2-5.2.

Procedure

Problem: To find a single value above which we may predict with confidence γ that a proportion P of the population will lie.

(1) Choose P the proportion and γ, the confidence coefficient.

(2) Compute:
$\bar{X}$
s

(3) Look up K in Table A-7 for the appropriate n, γ, and P.

(4) $X_L = \bar{X} - Ks$

Example

Problem: To find a single value above which we may predict with 90% confidence that 99% of the population will lie. (Data Sample 2-1).

(1) Let $P = .99$
$\gamma = .90$

(2)
$\bar{X} = .1260$ inch
$s = 0.00359$ inch

(3) $K\ (10, .90, .99) = 3.532$

(4) $X_L = .1260 - 3.532\ (.00359)$
$= .1133$ inch

Thus we are 90% confident that 99% of the mica washers will have thicknesses above .113 inch.

Note: Factors for some values of n, γ, and P not covered in Table A-7 may be found in Sandia Corporation Monograph SCR-13[2]. Alternatively, one may compute K using the following formulas:

$$a = 1 - \frac{z_\gamma^2}{2(n-1)} \quad \text{(where } z \text{ can be found in Table A-2)}$$

$$b = z_P^2 - \frac{z_\gamma^2}{n}$$

$$K = \frac{z_P + \sqrt{z_P^2 - ab}}{a}$$

2-5.4 TOLERANCE LIMITS WHICH ARE INDEPENDENT OF THE FORM OF THE DISTRIBUTION

The methods given in Paragraphs 2-5.2 and 2-5.3 are based on the assumption that the observations come from a normal distribution. If the distribution is not in fact normal, then the effect will be that the true proportion P of the population between the tolerance limits will vary from the intended P by an amount depending on the amount of departure from normality. If the departure from normality is more than slight we can use a procedure which assumes only that the distribution has no discontinuities. The tolerance limits so obtained will be substantially wider than those assuming normality.

2-5.4.1 Two-Sided Tolerance Limits (Distribution-Free)

Table A-30 gives values (r, s) such that we may assert with confidence at least γ that $100P\%$ of a population lies between the r^{th} smallest and the s^{th} largest of a random sample of n from that population. For example, from Table A-30 with $\gamma = .95$, $P = .75$, and $n = 60$, we may say that if we have a sample of $n = 60$, then we may have a confidence of at least $\gamma = .95$ that $100P\% = 75\%$ of the population will lie between the fifth largest $(s = 5)$ and the fifth smallest $(r = 5)$ of the sample values. That is, if we were to take many random samples of 60, and take the fifth largest and fifth smallest of each, we should expect to find that at least 95% of the resulting intervals would contain 75% of the population.

Table A-32 may be useful for sample sizes of $n \leq 100$. This table gives the confidence γ with which we may assert that $100P\%$ of the population lies between the largest and smallest values of the sample.

2-5.4.2 One-Sided Tolerance Litmits (Distribution-Free)

Table A-31 gives the largest value of m such that we may assert with confidence at least γ that $100P\%$ of a population lies below the m^{th} largest (or above the m^{th} smallest) of a random sample of n from that population. For example, from Table A-31 with $\gamma = .95$, $P = .90$, and $n = 90$, we may say that we are 95% confident that 90% of a population will lie below the fifth largest value of a sample of size $n = 90$.

REFERENCES

1. M. G. Kendall and W. R. Buckland, *A Dictionary of Statistical Terms,* p. 79, Oliver and Boyd, London, 1957.
2. D. B. Owen, *Table of Factors for One-Sided Tolerance Limits for a Normal Distribution,* Sandia Corporation Monograph SCR-13, April 1958.

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TABLE A-7. FACTORS FOR ONE-SIDED TOLERANCE LIMITS FOR NORMAL DISTRIBUTIONS

Factors K such that the probability is γ that at least a proportion P of the distribution will be less than $\overline{X} + Ks$ (or greater than $\overline{X} - Ks$), where $\overline{X}$ and s are estimates of the mean and the standard deviation computed from a sample size of n. see p 2-14

	$\gamma = 0.75$					$\gamma = 0.90$				
n \ P	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
3	1.464	2.501	3.152	4.396	5.805	2.602	4.258	5.310	7.340	9.651
4	1.256	2.134	2.680	3.726	4.910	1.972	3.187	3.957	5.437	7.128
5	1.152	1.961	2.463	3.421	4.507	1.698	2.742	3.400	4.666	6.112
6	1.087	1.860	2.336	3.243	4.273	1.540	2.494	3.091	4.242	5.556
7	1.043	1.791	2.250	3.126	4.118	1.435	2.333	2.894	3.972	5.201
8	1.010	1.740	2.190	3.042	4.008	1.360	2.219	2.755	3.783	4.955
9	0.984	1.702	2.141	2.977	3.924	1.302	2.133	2.649	3.641	4.772
10	0.964	1.671	2.103	2.927	3.858	1.257	2.065	2.568	3.532	4.629
11	0.947	1.646	2.073	2.885	3.804	1.219	2.012	2.503	3.444	4.515
12	0.933	1.624	2.048	2.851	3.760	1.188	1.966	2.448	3.371	4.420
13	0.919	1.606	2.026	2.822	3.722	1.162	1.928	2.403	3.310	4.341
14	0.909	1.591	2.007	2.796	3.690	1.139	1.895	2.363	3.257	4.274
15	0.899	1.577	1.991	2.776	3.661	1.119	1.866	2.329	3.212	4.215
16	0.891	1.566	1.977	2.756	3.637	1.101	1.842	2.299	3.172	4.164
17	0.883	1.554	1.964	2.739	3.615	1.085	1.820	2.272	3.136	4.118
18	0.876	1.544	1.951	2.723	3.595	1.071	1.800	2.249	3.106	4.078
19	0.870	1.536	1.942	2.710	3.577	1.058	1.781	2.228	3.078	4.041
20	0.865	1.528	1.933	2.697	3.561	1.046	1.765	2.208	3.052	4.009
21	0.859	1.520	1.923	2.686	3.545	1.035	1.750	2.190	3.028	3.979
22	0.854	1.514	1.916	2.675	3.532	1.025	1.736	2.174	3.007	3.952
23	0.849	1.508	1.907	2.665	3.520	1.016	1.724	2.159	2.987	3.927
24	0.845	1.502	1.901	2.656	3.509	1.007	1.712	2.145	2.969	3.904
25	0.842	1.496	1.895	2.647	3.497	0.999	1.702	2.132	2.952	3.882
30	0.825	1.475	1.869	2.613	3.454	0.966	1.657	2.080	2.884	3.794
35	0.812	1.458	1.849	2.588	3.421	0.942	1.623	2.041	2.833	3.730
40	0.803	1.445	1.834	2.568	3.395	0.923	1.598	2.010	2.793	3.679
45	0.795	1.435	1.821	2.552	3.375	0.908	1.577	1.986	2.762	3.638
50	0.788	1.426	1.811	2.538	3.358	0.894	1.560	1.965	2.735	3.604

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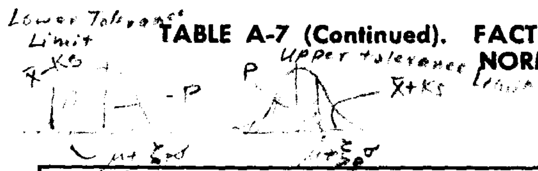
TABLE A-7 (Continued). FACTORS FOR ONE-SIDED TOLERANCE LIMITS FOR NORMAL DISTRIBUTIONS



n \ P	γ = 0.95 0.75	0.90	0.95	0.99	0.999	γ = 0.99 0.75	0.90	0.95	0.99	0.999
3	3.804	6.158	7.655	10.552	13.857	—	—	—	—	—
4	2.619	4.163	5.145	7.042	9.215	—	—	—	—	—
5	2.149	3.407	4.202	5.741	7.501	—	—	—	—	—
6	1.895	3.006	3.707	5.062	6.612	2.849	4.408	5.409	7.334	9.540
7	1.732	2.755	3.399	4.641	6.061	2.490	3.856	4.730	6.411	8.348
8	1.617	2.582	3.188	4.353	5.686	2.252	3.496	4.287	5.811	7.566
9	1.532	2.454	3.031	4.143	5.414	2.085	3.242	3.971	5.389	7.014
10	1.465	2.355	2.911	3.981	5.203	1.954	3.048	3.739	5.075	6.603
11	1.411	2.275	2.815	3.852	5.036	1.854	2.897	3.557	4.828	6.284
12	1.366	2.210	2.736	3.747	4.900	1.771	2.773	3.410	4.633	6.032
13	1.329	2.155	2.670	3.659	4.787	1.702	2.677	3.290	4.472	5.826
14	1.296	2.108	2.614	3.585	4.690	1.645	2.592	3.189	4.336	5.651
15	1.268	2.068	2.566	3.520	4.607	1.596	2.521	3.102	4.224	5.507
16	1.242	2.032	2.523	3.463	4.534	1.553	2.458	3.028	4.124	5.374
17	1.220	2.001	2.486	3.415	4.471	1.514	2.405	2.962	4.038	5.268
18	1.200	1.974	2.453	3.370	4.415	1.481	2.357	2.906	3.961	5.167
19	1.183	1.949	2.423	3.331	4.364	1.450	2.315	2.855	3.893	5.078
20	1.167	1.926	2.396	3.295	4.319	1.424	2.275	2.807	3.832	5.003
21	1.152	1.905	2.371	3.262	4.276	1.397	2.241	2.768	3.776	4.932
22	1.138	1.887	2.350	3.233	4.238	1.376	2.208	2.729	3.727	4.866
23	1.126	1.869	2.329	3.206	4.204	1.355	2.179	2.693	3.680	4.806
24	1.114	1.853	2.309	3.181	4.171	1.336	2.154	2.663	3.638	4.755
25	1.103	1.838	2.292	3.158	4.143	1.319	2.129	2.632	3.601	4.706
30	1.059	1.778	2.220	3.064	4.022	1.249	2.029	2.516	3.446	4.508
35	1.025	1.732	2.166	2.994	3.934	1.195	1.957	2.431	3.334	4.364
40	0.999	1.697	2.126	2.941	3.866	1.154	1.902	2.365	3.250	4.255
45	0.978	1.669	2.092	2.897	3.811	1.122	1.857	2.313	3.181	4.168
50	0.961	1.646	2.065	2.863	3.766	1.096	1.821	2.296	3.124	4.096